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GROUND WATER EXCHANGE WITH FLUCTUATING RIVERS*

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INTRODUCTION

Behavior of ground water adjacent to fluctuating rivers, lakes or reservoirs is of interest to the hydrologist for several reasons. A surface water body may derive most of its flow from ground water sources in some instances.

In other cases, the ground water can be almost totally composed of water from the surface body. These two conditions (or a variety of modifications) may exist in the same region at different times, depending on the stage of the surface and ground waters. The changing elevation of the surface water may cause changes in ground water quality and quantity available for use due to inflow or outflow. Or, conversely, the fluctuations may cause similar changes in the surface water characteristics from ground water discharge.

Knowledge of ground water exchange with varying stage surface water can lead to better and more efficient use, conservation and control of both ground and surface waters. Much research is being done in the field of hydrology to study these phenomena, particularly as they relate to the return flow of ground water to a river following passage of a flood stage.

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This paper will present some background information on hydraulics of water flow adjacent to a fluctuating river, discuss evaluation methods, and show how these methods were used for field problem solution on the Hanford Project.

SUMMARY AND CONCLUSIONS

Information is presented on water behavior adjacent to fluctuating rivers concerning bank storage and river water penetration into ground water aquifers. Methods of investigation are discussed and bank storage and river water exchange evaluation is made for an area where a large amount of geologic and hydrologic data are available.

Finally, evaluation of storage and exchange is carried out for the Hanford Project side of the Columbia River. Total bank storage for a typical year was 2.0 x 109 cubic feet of which 36 per cent was river water. Total river water in the aquifer was

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2.9 x 10^9 cubic feet. This volume is equivalent to only about eight hours river flow, based on an average flow rate of 1 x 10^5 ft³/sec.

Methods which exist for qualitative and semi-quantative evaluation of ground water exchange with fluctuating surface water bodies were found to be applicable even if only minimal hydrologic data are available. Such evaluations contribute to the overall knowledge of area hydrology and lead to effective and efficient use of both ground and surface waters.

DISCUSSION

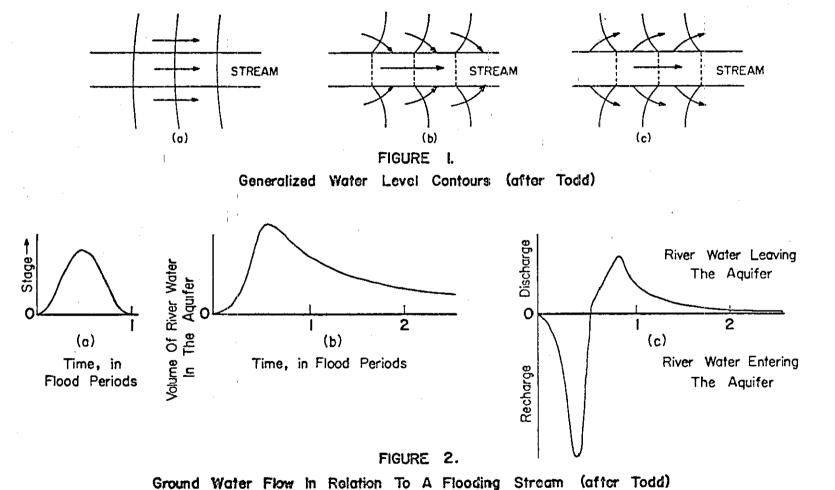
Hydraulics of Water Flow Adjacent to a Fluctuating River

Figure 1 (after Todd) (2) shows generalized water level contours under three different stream conditions. (a) Shows a "normal" stream where surface water and ground water stages are equal and no water exchange occurs. (b) Shows an effluent stream where the river stage is lower than the ground water stage, and ground water is flowing into the river. (c) Shows an influent stream where the surface water stage is higher than ground water elevation and the river is discharging water into the aquifer.

Figure 2 (after Todd) (2) shows idealized ground water flow in relation to a flooding stream. (a) Shows the stream hydrograph, stage plotted against flood period. (b) Shows that the volume of river water in aquifer increases as the river rises, reaches a maximum at about crest stage and then gradually decreases with time. (c) Shows the ground water flow curve which is the derivative of the volume curve.

Figure 3 shows a vertical cross section of an aquifer and adjacent river; b is the aquifer thickness. The base flow is ground water that flows to the river at initial water table and river stage. If the river level rises by a distance h_0 , the water table rises due to the inflow of river water and at time t rises by a height of h at x distance back from the river. Therefore, we see that a stream fluctuation may produce large variations in magnitude and direction of ground water flow.

Bank storage is the general term used for the river water stored in the aquifer during flood stage. In this paper, however, bank storage is defined as water, both river and ground that is stored in the zone above base flow stage. This is depicted at flood stage by the crosshatched zone in Figure 3. At low stage, this water is contributed to the river in addition to the "exchanged" river water in the base flow zone and thus is truly stored water.



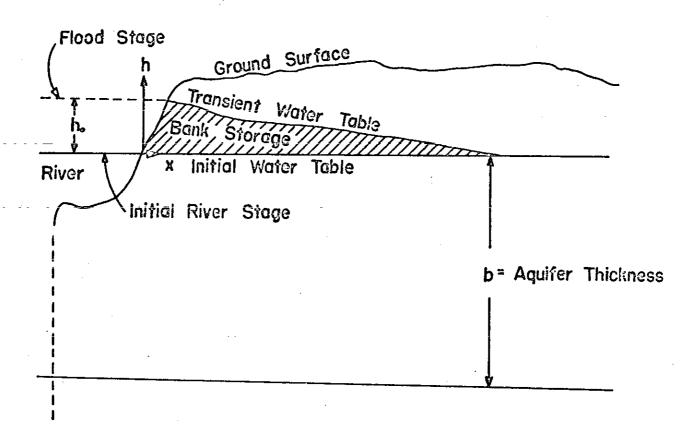


FIGURE 3.
Aquifer And River Cross Section

Methods and Techniques for Investigating Bank Storage and Exchange Water Behavior

It is evident that much information on the geology and hydrology of a region is needed for good quantitative evaluation of bank storage and water exchange. Ideally, we would like river hydrographs covering several flood periods and well hydrographs at varying distances from the river for the same periods, accurate values for aquifer coefficients of permeability and storage, and detailed geologic information for delineation of aquifer boundaries.

Where little information is available, river volume or flow rate can be measured at both ends of a given reach of river. The downstream value can then be corrected for rainfall contribution, surface water entry or discharge and evapotranspiration. Differences between the upstream and downstream values represent volume of base flow, bank storage and exchange or bank discharge depending on river stage.

This would be a very rough estimate except in those cases where aquifer inflow and outflow is a significant portion of the stream flow. At Hanford, we are fortunate in having a large amount of information on the area geology and hydrology. This has assisted greatly in assessing rate and direction of ground water movement, river and ground water exchange and aquifer coefficients.

Figure 4 shows the general location of the Hanford Project. To the north the Wahluke Slope area is situated between the Columbia River and the Saddle Mountains. The land area to the east of Hanford is part of the Columbia Irrigation Project. This land block rises abruptly from the-river level of about 350 feet to over 900 feet. Several coulees dissect this land block. At the southern end of the Columbia Irrigation Project is the City of Pasco. The southern boundary of the Hanford Project is delineated by the City of Richland, the Yakima River and the Rattlesnake Hills. The upland area comprising the Yakima Ridge and Umtanum Ridge lies to the west. The location of the 300 Area is shown in Figure 4. An example of river-ground water exchange evaluation will be presented, based on data from this site.

The project area is underlain by basaltic lavas and sedimentary interbeds of the Columbia River basalt series. These volcanics are overlain by a series of poorly consolidated lacustrine gravels, sands, silts, and clays known as the Ringold formation. The Ringold Formation is overlain by recent fluviatile and glaciofluviatile sands and gravels.

Contours on the ground water surface (Figure 5) show that the

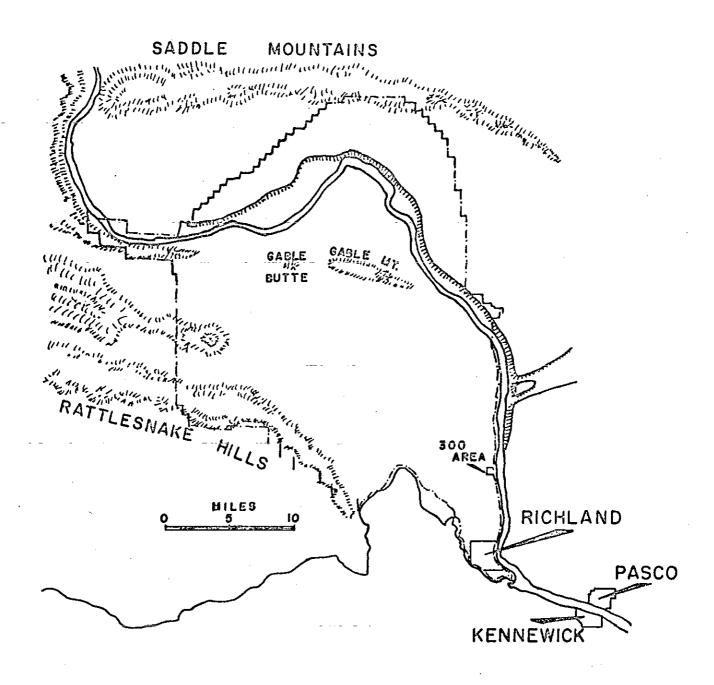


FIGURE 4. Hanford Reservation And Vicinity

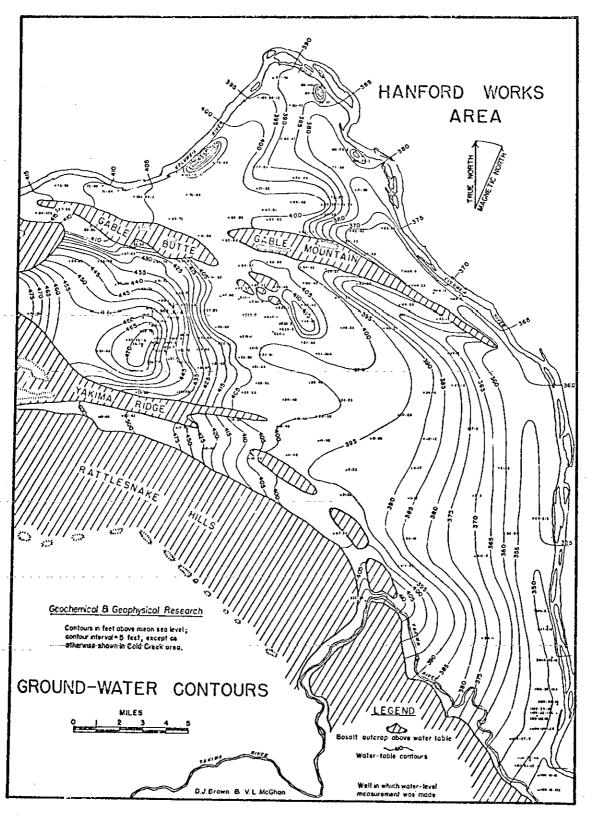


FIGURE 5.

ground water originates from the mountainous areas of recharge to the southwest and moves in a general eastward direction to the Columbia River. Appreciable modification occurs in the local and regional water table due to ground disposal of large volumes of process cooling water.

The 300 Area was chosen as an example of bank storage and water exchange evaluation because of the high well density (19 in an area of about one square mile), presence of small amounts of uranium in the ground water, and the availability of well and river hydrographic data over a period of several years.

The well distances to the Columbia River vary from 50 feet to 4,000 feet and permit a fairly complete geologic and hydrologic evaluation of the area. The uranium contamination in the ground water permits tracing of water flow. Figure 6 shows hydrographs of the river and two wells during flood stage for a typical year (1950).

Also shown are the water temperatures of the two wells. The river (solid line) begins to rise from a base of 334 feet m.s.l., during the latter part of March, increasing to about 340 feet in the middle of May and rapidly rises to flood stage of 356 feet on June 26. Well No. 6 (dashed line) is 600 feet from the river and rises almost linearly to a stage of 354.5 feet on July 7.

Well No. 8 (dotted) is 4,000 feet from the river and rises linearly to a stage of 354 feet on July 11. Base elevations of wells 6 and 8 are about 336 feet and 338 feet, respectively. Amplitudes of the river, well 6 and well 8 above base at flood stage are 24, 18.5 and 16 feet.

At low stage the river has a flow rate of about 6.0 x 10^4 ft³/sec., and at flood stage has a flow of about 5.2 x 10^5 ft³/sec. Aquifer thickness in the 300 Area averages about 70 feet (effective) and most of the flow occurs in the glaciofluviatile deposits.

As the river water rises, the temperature gradually increases and reaches 53F at flood. Normal ground water temperature remains fairly constant throughout the year between 59F to 62F. Therefore, presence of gross quantities of river water in the aquifer should be easily detected by temperature differences. Water temperature in well No. 6 during flood period is indicated by the solid line in Figure 6.

Presence of river water is indicated by the temperature decrease from about 57.5F on May 15 to about 52F on July 10. The temperature increases again as the river falls and river water flows out of the aguifer. The dashed line shows water temperature in

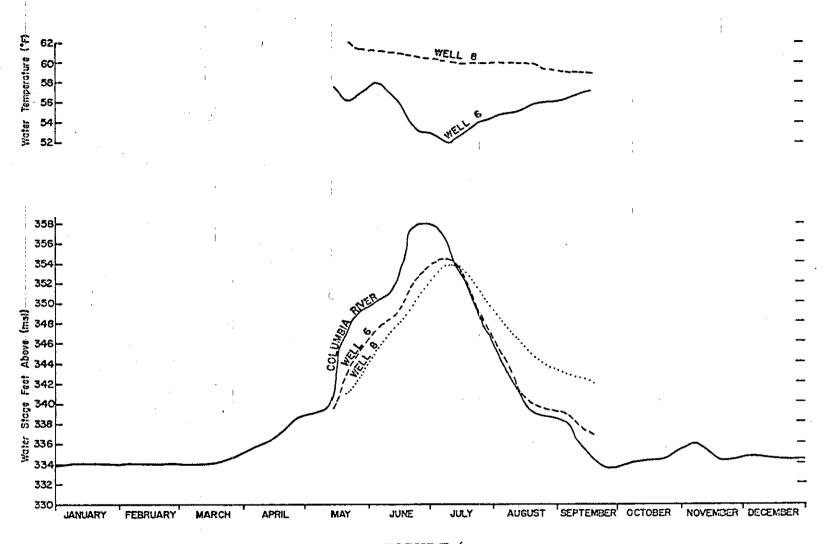


FIGURE 6.
300 Area Hydrographs And Well Water Temperature

well No. 8 during the same period. The temperature only changes from 62F to 59F, indicating that river water is not present in this well.

Figure 7 shows profiles of ground water elevation at low stage and at flood stage in the 300 Area. Also shown are temperature profiles at flood and low stages. Liquid wastes that contain small amounts of natural uranium were disposed of at_this__site, about 600 feet back from the river. The profiles of this ground water contamination (alpha d/m/liter) are shown for low stage, flood stage and mid-stage during ebb flow. Uranium contaminated water was pushed back from entry point at 600 feet from the river to about 2700 feet by river water inflow. At mid-stage ebb flow, the peak has moved toward the river.

At low stage, a residual peak remains at about 2400 feet, but the ground water flow is diluting the inflow of contaminated water and carrying it to the river. The temperature profile at flood stage shows that river water has penetrated to a distance of about 2000 feet from the river bank. The wedge shaped area between the two ground water stages is bank storage. The vertical dashed line shows the approximate extent of river water infiltration.

Volume of bank storage and infiltered river water at flood stage may now be estimated from the available field data. Bank storage was found to be 1.08×10^5 square feet or 5.7×10^8 ft³/mile of river bank. This entire volume, of course, is not water.

The water only occupies the pore space in the aquifer. The volume available for water transmission is called effective porosity or, for water table conditions, the numerically equivalent storage coefficient. The coefficient of storage is defined as the volume of water released from storage from one square foot of aquifer when the hydraulic head is reduced one foot. The storage coefficient can be determined from pumping tests. Effective porosity can also be determined from the relationship: S = f(Q/AV) where S is the effective porosity (or storage coefficient), V the average ground water velocity, Q the total quantity of flow through cross Section A, and f is a temperature correction factor used when the water temperature varies from 60F.

In use on the project, V was obtained from tracer tests using radioactive isotopes, Q was a measured amount of cooling water discharged to the ground and the A was determined by temperature measurements. S can vary from about 0.05 to 0.3 under water table conditions. S for the 300 Area was found to be about 0.1, indicating that about one tenth of the volume was active in transmitting and storing water.

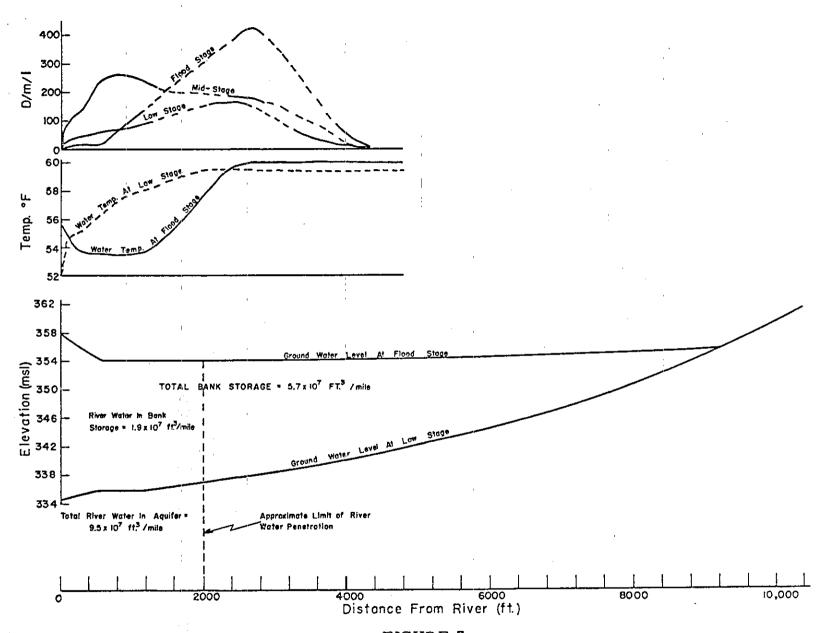


FIGURE 7.
Ground Water Elevation, Contamination And Temperature Profiles

Therefore, the bank storage zone equals 5.7×10^8 cubic feet times 0.1 or 5.7×10^7 cubic feet of water stored per mile of river. The indicated river infiltration zone within the bank storage region has an area of about 3.6×10^4 square feet or 1.90×10^8 ft³/mile, which multiplied by the storage coefficient equals 1.90×10^7 cubic feet of river water in bank storage per mile.

About 30 per cent of the total bank storage is river water and the remaining 70 per cent is ground water. The total volume of infiltrated river water can be estimated by determining the volume of the infiltrated zone per mile (distance of infiltration times effective aquifer thickness times 5280 feet), which is 9.5×10^8 cubic feet multiplied by the storage coefficient equals 9.5×10^7 cubic feet of river water per mile of bank.

The same problem may also be evaluated by use of the equations noted in Figure 8. The aquifer coefficients of transmissibility (or permeability and aquifer thickness) and storage must be known. Evaluation of the coefficient of storage was discussed above. The aquifer transmissibility, the ability of the porous media to transmit water, can be determined by pumping tests, tracer tests and well water fluctuation observation. The latter method is particularly adapted for use near the fluctuating streams. Methods developed by Werner and Noren (5), Ferris (6) or Rowe (7) may be used.

A number of assumptions were made to simplify derivation of equations (1) and (2). These conditions often are not found in field situations; therefore, use of these equations will provide only an estimate of storage and exchange. Reference (4) in the bibliography gives the complete derivation of the equations. The values of 1 - P(x) are obtained from a tabulation of the probability integral, P(x), and are plotted against $x/\sqrt[3]{1}$. α is calculated from observed field data. The values of 1 - P(x) and thus h/h_0 are determined from values of $x/\sqrt[3]{1}$. Table I shows comparison of transmissibility values calculated by several different methods for well No. 8, 4,000 feet from the river. The T value calculated by the time lag and amplitude ratio methods, about 1.2 x 10^6 gallons per day per foot (g/d/ft), is used in the examples that follow.

If the change in river stage h_{o} is known, water table rise h can be determined at any time t for any distance x from the river. Equation 2 can be used to estimate the total volume of river water V in the aquifer at time t for a river stage rise of h_{o} . Equation 1 was used to calculate the water table rise above river base level 4,000 feet from the river at flood stage $(h_{o} = 24 \text{ feet})$.

An assumption was made in equation derivation that the river

Where h = water table rise at distance x from river at time t since river stage was raised a distance ho

P(x) is the error function erf $(x/\sqrt{4\mu t})$ sometimes called the probability integral, given by:

$$P(x) = 2/\sqrt{n} \int_{0}^{x/\sqrt{n}} e^{-u^{2}} du$$

K = Coefficient of permeability

b = Aquifer thickness

S = Coefficient of storage

T = Kb

$$V = \frac{2Kh_0b}{\sqrt{n_{\alpha}}} \sqrt{t}$$
 - - - - (2).

Where V = Total volume of river water in the aquifer at time t.

FIGURE 8

EQUATIONS FOR EVALUATING BANK STORAGE AND RIVER WATER EXCHANGE

stage changes instantaneously. This is not true for the example problem where the stage change ho takes place over a period of 90 days. A better approximation is made if the stage change is broken into small increments that occur over a relatively short time. h is then calculated for each incremental rise and the individual h values summed to give total aquifer rise above base. Table II shows the results of these calculations. Total calculated rise is 21.5 feet (flood stage plus 11 days total time t=101 days) or an elevation of 355.5 feet, m.s.l., 4,000 feet from the river at flood stage, plus 11 days. Observed water table elevation at the well 4,000 feet from the river is 354 feet at flood stage. The calculated value is only 1.5 feet higher than the actual observation.

Equation 2 was used to estimate the total volume of river water in the aquifer at flood stage (t = 90 days). The same incremental stage change technique was used for evaluation. Table III shows the results of these calculations. Calculated total river water in the aquifer at flood stage is about 1.08 x 10^8 ft³/mile of river bank. Observed river water in the aquifer was 9.5 x 10^7 ft³/mile. The calculated value is 1.14 times the observed value, which is considered good agreement.

Bank Storage and River Exchange on the Hanford Project

To evaluate the bank storage and river water exchange throughout the entire reach of the Hanford Project it is first necessary to consider the geohydrologic changes which occur along this section of the river. Studies have shown that there are four distinct regions which differ significantly in aquifer characteristics and should, therefore, be evaluated individually.

Figure 9 is a diagrammatic sketch showing the reach of the Columbia River from Richland to Priest Rapids, with cross sections depicting the major geologic changes in the zone where bank storage occurs. This figure shows the two relative positions of the river stage which define the bank storage zone.

The lower stage (represented by a solid line) is based on an average minimum flow rate of 50,000 ft³/second while the upper stage (represented by the dashed line) is based on an average maximum flow rate of 500,000 ft³/sec. The geology of particular interest in this discussion is limited to the rocks through which the ground water percolates at these two extreme stages. The subdivision into regions is indirectly based on the geologic changes which are shown in this diagram.

Within Region I the rocks are predominatly sands and gravels of the glaciofluviatile deposit. These rather coarse materials range in size from boulders, several feet in diameter to fine

TABLE I

300 Area Transmissibility Values

Method

T values (g/d/ft) Well No. 8

Rowe	Ferris (Time lag)	Ferris (Amp. ratio)	Pumping Test
1.6 x 10 ⁶	1.19 x 10 ⁶	1.16 x 10 ⁶	1.5 x 10 ⁶

TABLE II

300 Area Water Table Rise at Flood Stage

$$h/h_o = erfc (x/\sqrt{4\alpha t})$$

 $h/h_o = 1 - P(x)$

$$\alpha = T/S - 1.2 \times 10^7$$
 $4\alpha = 4.8 \times 10^7$
 $S = 0.1$
 $x = 4.000 \text{ ft.}$

h _o (feet)	t (days)	$x/\sqrt{4\alpha t}$	h/ho	h
2	101	0.06	0.93	1.86
2	- 83	0.0635	0.92	1.84
2	70	0.069	0.91	1.82
6	55	0.078	0.91	5.45
2 .	51	0.078	0.91	1.82
2	43	0.087	0.90	1.80
2	30	1.1055	0.88	1.76
2	24	0.118	0.86	1.72
2	. 22	0.118	0.86	1.72
1	21	0.1215	0.86	0.86
<u> </u>	16	0.158	0.82	0.82
Total 24				21.5 -

21.5 - Ft. water table rise above river base, 4,000 feet from river at flood stage.

Observed rise above = 20.0 Ft. river base Difference = 1.5 Ft.

TABLE III

River Water in the Aquifer at Flood Stage $V = \frac{(5280 (2 T h_0))}{\sqrt{\pi \alpha}} \sqrt{t}$

$$T = 1.2 \times 10^6$$

 $\alpha = 1.2 \times 10^7$ g/d/ft

h _o (Ft)	t (days)	\sqrt{t}	Volume	
2	90	9.49	1.47 x 10 ⁷ Ft ³ /	_
2	81	9.00	$1.362 \times 10^{7} \text{ Ft}$	
2 .	58	7.61	$1.153 \times 10^{7} \text{ Ft}$	
6	43	6.56	<u> </u>	3/mile
2	39	6.24		3/mile
2	31	5.57		3/mile
2	18	4.24		3/mile
2	12	3.46		3/mile
2	10	3.16		3/mile
1	9	3.00		3/mile
1	4	2.00	1.52 x 10 ⁶ Ft	3/mile

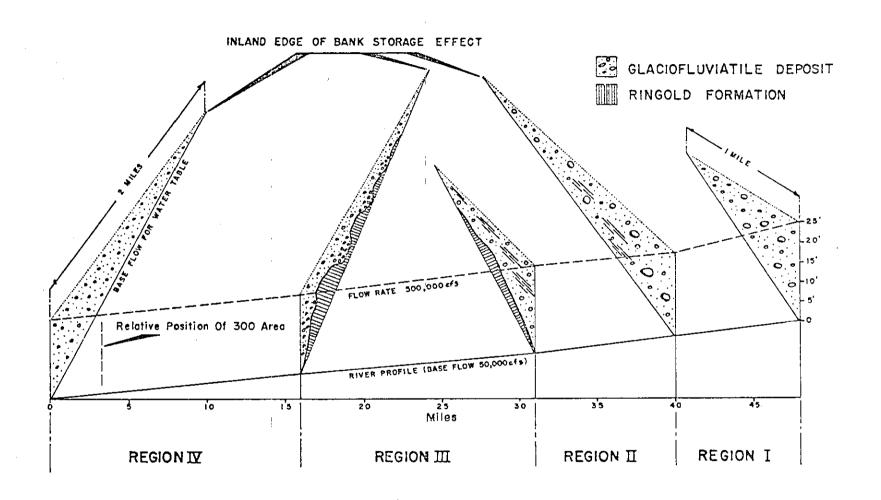


FIGURE 9. Regional Geologic Changes In Bank Storage Zone

sand, only a few thousandths of an inch in diameter. A gradual facies change occurs between Region I and Region II within the glaciofluviatile deposit. The amount of silt gradually increases from Region I to Region II. In addition to the increased amount of silt in the glaciofluviatile deposit, sediments of the Ringold Formation are present in the lower portion of this zone.

These materials are predominantly made up of cemented sand and gravel. Again a change occurs between Regions II and III. The Ringold beds underlying the glaciofluviatile deposit in Region III are silt and clay size sediments rather than the cemented material found in Region II. The amount of silt in the glaciofluviatile deposit diminishes in Region III to that more like Region I, however the sands and gravels show more evidence of sorting.

Region IV, being the farthest downstream, shows the greatest amount of sorting. All of the sediments in Region IV are glacio-fluviatile in origin. The evaluation of the 300 Area bank storage reflects the uniformity of the sediments found in this region. The relative position of the 300 Area is also shown in Figure 9.

The bank storage and river water exchange was determined in each of these four regions using the same evaluation techniques such as were discussed in the 300 Area case. For example, Figure 10 shows the river and well hydrographs which were used to calculate the transmissibility for Region I. These hydrographs dramatically point out the degree to which the Ground Water Table fluctuates in response to the change in river stage. Sufficient information was obtained from these data to calculate the transmissibility for this region by the Rowe method, and by the timelag and stage-ratio methods of Ferris. Similar data are available and were used for all of the four regions.

Figure 11 is a map of the Hanford Project showing the inland boundary of the ground water fluctuation and the portion of the zone, delineated by the temperature survey, believed to be direct river water recharge. Beyond the limit of the hachured line running roughly parallel to the river, there is no evidence of water table fluctuation that can be attributed to the stage changes of the Columbia River.

The enclosed table in Figure 11 shows the aquifer characteristics of each of the four regions and those values used to determine the total bank storage. A breakdown of this total bank storage for the Hanford Project appears in Table IV by regions. It appears significant that of the total 2.0 x 10⁹ cubic feet of water in bank storage on the Project side of the river, only 36 per cent is actual river water.

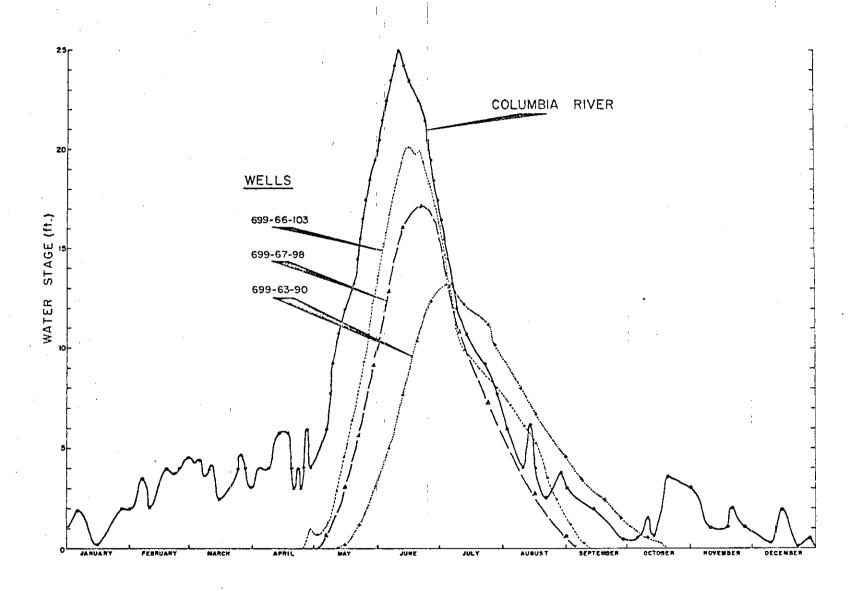
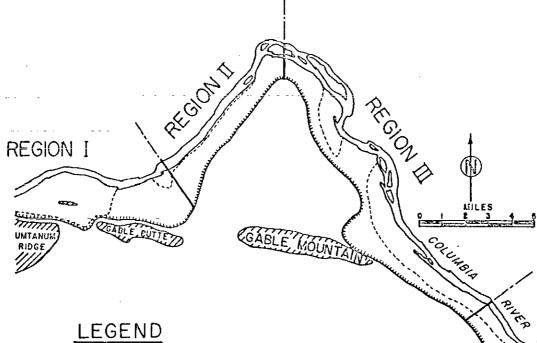


FIGURE 10. Region I River And Well Hydrographs



Inland extent of water-table fluctuations resulting from changes in river stage.

Inland limit of river water movement as defined from temperature profiles.

REGIONAL	AQUIF	ER CH	ARACTE	RISTICS
				REGION IV
Transmissibility g/d/ft	2.0 X-10 ⁶ -	1.0 X 10 ⁴	-4.6 X 10 ⁴	1.2 × 10 ⁶
Storage Coefficient	0.05	0.01	0.1	0.1
Aquifer Thickness ft	70	25	10	70
Length of Aquifer miles	8	9	15	16
Permeability - =	3.0 x 10 4 -	4.70 x 10.2	4.6 X 10 ³ -	1,7 °x 16 ⁴ · · · ·

FIGURE 11. - Inland Boundary-Of-Ground Water Fluctuation And River Water Penetration

In 1961 the U. S. Geological Survey made a reconnaissance evaluation of bank storage along this same reach of the Columbia River for the Atomic Energy Commission (8). They determined the total bank storage to be 3.4 x 10 oubic feet, which is in good agreement with that determined in this study; however, they concluded that approximately 99 per cent of the recharge was by river water.

It should be pointed out that the bank storage calculated for the Hanford Project side of the Columbia River cannot be used as an estimate for storage along the opposite bank of the river because of the completely different geohydrologic conditions present there.

ACKNOWLEDGMENT

The authors wish to thank L. G. King for his assistance in the preparation of this paper.

TABLE IV

Regional Sur	mmary of Bank	Storage	Beneath the	Hanford	Project
	R ¹ I	RII	R III	R IV	Total
Bank Storage (vol. in ft ³) 3.6x10 ⁸	2.8x10 ⁷	6.9x10 ⁸	9.1x10 ⁸	2.0x10 ⁹
Ground Water in Bank Stora (vol. in ft ³) % regional ba	age) 1.3x10 ⁸	2.3x10 ⁷	5.0x10 ⁸	6.1x10 ⁸	1.3x10 ⁹
storāge		98	72	67	64
River Water	in				, , ,
Bank Storage (vol. in ft ³ % of regional) 2.3x10 ²⁹	5x10 ⁶	1.9x10 ⁸	3x10 ⁸	7.3x10 ⁸
bank storage		2	28	33	36
Total River Water					
in Aquifer (vol. in ft ³) 9.27x10 ⁸	1.1x10 ⁸	3.8x10 ⁸	1.5x10 ⁹	2.9x10 ⁹

¹ R signifies Region.

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